

Finite difference schemes for two-dimensional miscible displacement flow in porous media on composite triangular grids[☆]

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Abstract

Considering two-dimensional miscible displacement flow in porous media, the local grid refinement method of a coupled system on triangular cell-centered grids with local refinement in space is studied. Based on the balance equation, finite difference schemes of the coupled equations on composite grids are constructed. Studying their stability and convergence properties, the error estimate in the energy norm is obtained. Finally, a numerical example is given.

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1. Introduction

In the process of petroleum exploitation, the pressure changes rapidly near wells. In order to improve the precision and save the cost of reservoir numerical simulations, we need to use the local grid refinement technique. Therefore it is very important to research the local grid refinement method of miscible displacement flow in porous media [1].

Finite volume (FV) method has a long history as a type of discretization tool for the numerical simulation of conservation laws. This method has been widely used in several engineering fields. The FV method based on cell-centered grids has been used by Pedrosa Jr. [2] for the efficient computation of fluid flow in porous media, and by Ewing, Lazarov and Vassilevski [3–5] for elliptic equations on rectangular grids with local refinement. Approximations of the convection–diffusion equation on rectangular composite grids are studied in [6]. Vassilevski, Petrova and Lazarov investigate approximations of second-order elliptic equations on triangular cell-centered grids with local refinement, and obtain error estimates in the discrete norm [7]. Cai and McCormick develop the FV element method for diffusion equations on vertex-centered and triangular composite grids [8]. Based on the FV approach, Ewing, Lazarov and Vassilevski construct difference schemes for parabolic problems on cell-centered rectangular grids with local refinement [9,10]. In [11], considering the localization phenomenon, the dynamic mixed finite element

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methods for incompressible miscible displacement in porous media are studied. Authors discuss the discretization of the semiconductor device problem on rectangular grids with local refinement in time and space in [12]. But until now there is little work on the local refinement method of the parabolic equation and the coupled equations on triangular grids. We can use a system of two nonlinear partial differential equations to describe the miscible displacement flow in porous media. The pressure equation is elliptic and the saturation equation is parabolic. In this paper, considering this mathematical model, we use the FV method to construct finite difference schemes of the coupled equations on triangular cell-centered grids with local refinement in space. The construction utilizes a modified upwind approximation and linear interpolation in space at the interface. Studying their stability and convergence properties, the error estimate in the energy norm is obtained. Finally, a numerical example is given to support the numerical method and its convergence.

The paper is organized as follows. In Sections 2 and 3 we formulate the problem and introduce composite grids. In Section 4 finite difference schemes on composite grids are derived. The convergence analysis is addressed in Section 5. Finally in Section 6 we present a numerical experiment. Throughout this paper, the symbol K is used to denote a generic constant.

2. Problem formulation

We consider a system of nonlinear partial differential equations in a bounded domain $\Omega \subset R^2$, which forms a basic model of miscible displacement flow in porous media [13]:

$$\begin{aligned} \text{(a)} \quad & \nabla \cdot u = q(x, t), \quad x \in \Omega, 0 < t \leq T, \\ \text{(b)} \quad & u = -a(x, c)\nabla p, \quad x \in \Omega, 0 < t \leq T, \\ \text{(c)} \quad & \phi(x) \frac{\partial c}{\partial t} + \nabla \cdot [uc - D\nabla c] = \bar{c}(x, t, c)q, \quad x \in \Omega, 0 < t \leq T. \end{aligned} \quad (2.1)$$

The pressure equation is elliptic and the saturation equation is parabolic. The unknowns are the pressure p , the Darcy velocity u and the saturation c . $a(x, c)$ is a term related to the permeability of the rock and the viscosity of the fluid; ϕ is the porosity of the rock; q is the external flow rate, which is positive where fluid is being injected and negative where fluid is being produced; \bar{c} is the externally-imposed saturation at an injection well and $\bar{c} = c$ at a production well. In this paper, we ignore the molecular dispersion, so $D = D(x) = \phi(x)d_{\text{mol}}I$.

In addition, we have initial and boundary conditions:

$$c(x, 0) = c_0(x), \quad x \in \Omega, \quad (2.2)$$

$$u \cdot \nu = 0, \quad c(x, t) = c^*(x, t), \quad (x, t) \in \partial\Omega \times (0, T], \quad (2.3)$$

where ν is the unit outer normal vector to $\partial\Omega$. Using conditions $\int_{\Omega} p dx = 0$ and $\int_{\Omega} q dx = 0$, where $0 \leq t \leq T$, we know the Eqs. (2.1)–(2.3) have unique solutions. We suppose that q is distributed smoothly and bounded; the exact solutions p and c satisfy

$$\frac{\partial c}{\partial t} \in L^2(H^2(\Omega)), \quad c \in L^\infty(H^3(\Omega)), \quad p \in L^\infty(H^3(\Omega)); \quad (2.4)$$

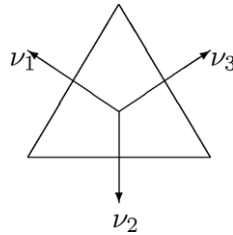
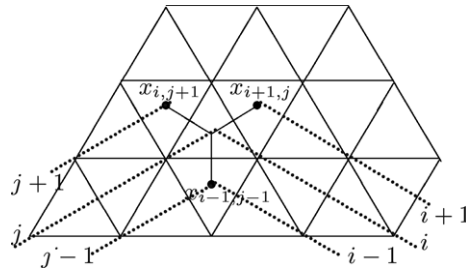
and the coefficients of (2.1) satisfy

$$0 < a_1 \leq a(x, c) \leq a_2, \quad 0 < D_1 \leq D(x) \leq D_2, \quad 0 < \phi_1 \leq \phi(x) \leq \phi_2, \quad (2.5)$$

where $a_1, a_2, D_1, D_2, \phi_1, \phi_2$ are constants, and $a(x, c)$ is Lipschitz continuous with respect to x and c .

3. Grids, grid functions and associated notations

We cover the region Ω by a set of triangles $\tilde{\tau} = \{e\}$. As shown in Fig. 1, let s_l ($l = 1, 2, 3$) be boundaries of e and ν_l ($l = 1, 2, 3$) be the unit outward normal vectors respectively to s_l . The interval $(0, T]$ is discretized using a regular grid with a parameter Δt , then discrete time levels $t^n = n\Delta t$, $n = 0, 1, \dots, J$, where $J = T/\Delta t$. We assume that all triangles in $\tilde{\tau}$ have no angles greater than or equal to $\frac{\pi}{2}$. Let the grid $\tilde{\omega}$ consist of the centers of the circumscribed circles of all the triangles $e \in \tilde{\omega}$. Here we consider only uniform partitions of Ω , by which we mean that every two

Fig. 1. Triangle $e \in \tau$.Fig. 2. Region Ω .

neighboring triangles form a parallelogram. As shown in Fig. 2, we assume further that the triangles $e \in \tilde{\tau}$ have sides parallel to three fixed directions, then we introduce a coordinate system using v_1 and v_3 as coordinate axes. Given the grid function $y(x, t) = y(x_1, x_2, t^n)$, where $t = t^n$, $x \in \tilde{\omega}$, we define

$$\Delta_l y^n(x) = y^n(x) - y^n(x + h_l v_l), \quad y_{x_l}^n = \frac{\Delta_l y^n(x)}{h_l}, \quad (3.1)$$

where h_l is the mesh size in the v_l direction ($l = 1, 2, 3$). Let $m(e) = \int_e ds$, we define the following discrete inner products and norms:

$$(y, z) = \sum_{x \in \omega} m(e) y(x) z(x), \quad \|y\|_{0,\omega}^2 = (y, y),$$

$$|y|_{1,\omega}^2 = \sum_{l=1}^3 \|y_{x_l}\|_{0,\omega}^2, \quad \|y\|_{1,\omega}^2 = \|y\|_{0,\omega}^2 + |y|_{1,\omega}^2.$$

For the case of local refinement, we refine some of the triangles in $\tilde{\tau}$ into four congruent triangles and suppose that the refinement is organized so that any nonrefined cell has at most one refined adjacent cell. We denote this new triangulation by τ . The circumcenters of triangles in τ form the composite grid ω . Let $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$, where Ω_2 is the refined region. The circumcenters of triangles in Ω_2 form the grid ω_2 , and let $\omega_1 = \omega \setminus \omega_2$.

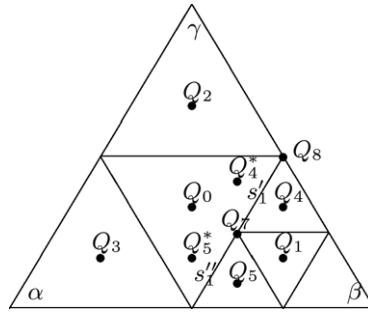
4. Construction of the finite difference schemes

Let P , U and C be the numerical approximations to p , u and c respectively. First we use the FV method to construct the discretization of the pressure equation (2.1)(a) on composite grids. For regular grids $x \in \omega_1$, let

$$\bar{w}_1^n(x) = \frac{m(s_1)}{h_1} (P_{i,j}^n - P_{i,j+1}^n) \bigg/ \frac{1}{h_1} \int_{x_{i,j}}^{x_{i,j+1}} \frac{ds}{a(s, C_{i,j+1/2}^n)} \equiv \bar{k}_1^n(x) \Delta_1 P^n(x),$$

$$\bar{w}_2^n(x) = \frac{m(s_2)}{h_2} (P_{i,j}^n - P_{i-1,j-1}^n) \bigg/ \frac{1}{h_2} \int_{x_{i,j}}^{x_{i-1,j-1}} \frac{ds}{a(s, C_{i-1/2,j-1/2}^n)} \equiv \bar{k}_2^n(x) \Delta_2 P^n(x),$$

$$\bar{w}_3^n(x) = \frac{m(s_3)}{h_3} (P_{i,j}^n - P_{i+1,j}^n) \bigg/ \frac{1}{h_3} \int_{x_{i,j}}^{x_{i+1,j}} \frac{ds}{a(s, C_{i+1/2,j}^n)} \equiv \bar{k}_3^n(x) \Delta_3 P^n(x), \quad (4.1)$$

Fig. 3. Interface triangle $e = e(x)$.

where $C_{i,j+1/2}^n = \frac{1}{2}(C_{i,j}^n + C_{i,j+1}^n)$, $C_{i-1/2,j-1/2}^n = \frac{1}{2}(C_{i,j}^n + C_{i-1,j-1}^n)$, $C_{i+1/2,j}^n = \frac{1}{2}(C_{i,j}^n + C_{i+1,j}^n)$, then we obtain the following cell-centered finite difference scheme of (2.1)(a):

$$\bar{w}_1^n(x) + \bar{w}_2^n(x) + \bar{w}_3^n(x) = \bar{\varphi}^n(x) = \int_{e(x)} q(\xi, t^n) d\xi. \quad (4.2)$$

Similarly for regular fine grids $x \in \omega_2$, we can also obtain the difference scheme (4.2).

We consider the interface triangle e , as shown in Fig. 3. Suppose that angles of e satisfy $\alpha = \beta = \gamma = \frac{\pi}{3}$, then let $h = h_1 = h_2 = h_3$, and $h = \begin{cases} h_c, & x \in \omega_1 \\ h_f, & x \in \omega_2 \end{cases}$, where h_c and h_f are mesh sizes of coarse and fine grids respectively. When we apply (4.2) into the element e , not all nodes are actual grids, such as Q_4^* and Q_5^* . For such cases, the values of the grid function at these nodes must be obtained by interpolation. Here we use piecewise linear interpolation in space, namely

$$\begin{aligned} P^n(Q_4^*) &= -\frac{1}{2}P^n(Q_3) + \frac{3}{2}P^n(Q_0), & P^n(Q_5^*) &= -\frac{1}{2}P^n(Q_2) + \frac{3}{2}P^n(Q_0), \\ C^n(Q_4^*) &= -\frac{1}{2}C^n(Q_3) + \frac{3}{2}C^n(Q_0), & C^n(Q_5^*) &= -\frac{1}{2}C^n(Q_2) + \frac{3}{2}C^n(Q_0). \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{w}_1^n(Q_4) &= \bar{k}_1^n(Q_4) \left[\Delta_1 P^n(Q_4) - \frac{1}{2} \Delta_3 P^n(Q_0) \right], \\ \bar{w}_1^n(Q_5) &= \bar{k}_1^n(Q_5) \left[\Delta_1 P^n(Q_5) - \frac{1}{2} \Delta_2 P^n(Q_0) \right], \\ \bar{w}_1^n(Q_0) &= \bar{w}_1^n(Q_4) + \bar{w}_1^n(Q_5), \end{aligned} \quad (4.3)$$

where

$$\bar{k}_1^n(Q_4) = \frac{m(s'_1)}{h_f} \left(\frac{1}{h_f} \int_{Q_4}^{Q_4^*} \frac{ds}{a(s, C_{\frac{Q_4+Q_4^*}{2}}^n)} \right)^{-1}, \quad \bar{k}_1^n(Q_5) = \frac{m(s''_1)}{h_f} \left(\frac{1}{h_f} \int_{Q_5}^{Q_5^*} \frac{ds}{a(s, C_{\frac{Q_5+Q_5^*}{2}}^n)} \right)^{-1}.$$

In this way, we have obtained the finite difference scheme of (2.1)(a) on composite grids ω .

The Darcy velocity $u = -a(x, c) \nabla p$ through the boundary of $e(x)$ is approximated as follows:

$$\begin{aligned} U_{1,i,j+1/2}^n &= -\frac{1}{2} [a(x_{i,j}, C_{i,j}^n) + a(x_{i,j+1}, C_{i,j+1}^n)] \cdot \frac{P_{i,j+1}^n - P_{i,j}^n}{h_1}, \\ U_{2,i-1/2,j-1/2}^n &= -\frac{1}{2} [a(x_{i,j}, C_{i,j}^n) + a(x_{i-1,j-1}, C_{i-1,j-1}^n)] \cdot \frac{P_{i-1,j-1}^n - P_{i,j}^n}{h_2}, \\ U_{3,i+1/2,j}^n &= -\frac{1}{2} [a(x_{i,j}, C_{i,j}^n) + a(x_{i+1,j}, C_{i+1,j}^n)] \cdot \frac{P_{i+1,j}^n - P_{i,j}^n}{h_3}. \end{aligned} \quad (4.4)$$

Next we consider the saturation equation (2.1)(c) on composite grids ω . For regular coarse grids $x \in \omega_1$ and the finite element $V = e(x) \times [t^n, t^{n+1}]$, we have

$$\int_{t^n}^{t^{n+1}} \int_e \phi(x) \frac{\partial c}{\partial t} dx + \int_{t^n}^{t^{n+1}} \int_{\partial e} v \cdot (-D(x) \nabla c + uc) ds dt = \int \int_V \bar{c}(x, t, c) q(x, t) dx dt, \quad (4.5)$$

where v is the unit outward normal to ∂e . Let e be the triangle shown in Fig. 1, then $\partial e = s_1 \cup s_2 \cup s_3$. Let $W_v = -v \cdot D \nabla c$ and $V_v = v \cdot uc$, we can obtain

$$\int_{t^n}^{t^{n+1}} \int_e \phi(x) \frac{\partial c}{\partial t} dx + \int_{t^n}^{t^{n+1}} \left[\sum_{l=1}^3 \left(\int_{s_l} W_{v_l} ds_l + \int_{s_l} V_{v_l} ds_l \right) \right] dt = \int \int_V \bar{c}(x, t, c) q(x, t) dx dt. \quad (4.6)$$

For every element $V = V(x, t^{n+1})$, we denote by $\Delta t w_l^{n+1}$ and $\Delta t v_l^{n+1}$ the approximations respectively to $\int_{t^n}^{t^{n+1}} \int_{s_l} W_{v_l} ds_l dt$ and $\int_{t^n}^{t^{n+1}} \int_{s_l} V_{v_l} ds_l dt$, where $l = 1, 2, 3$. For regular coarse grids, let

$$\begin{aligned} w_l^{n+1}(x) &= \frac{k_l(x)}{1 + |B_l^n(x)|/k_l(x)} \Delta_l C^{n+1}(x), \quad l = 1, 2, 3, \\ v_1^{n+1}(x) &= (B_1^n(x) + |B_1^n(x)|) C_{i,j}^{n+1} + (B_1^n(x) - |B_1^n(x)|) C_{i,j+1}^{n+1}, \\ v_2^{n+1}(x) &= (B_2^n(x) + |B_2^n(x)|) C_{i,j}^{n+1} + (B_2^n(x) - |B_2^n(x)|) C_{i-1,j-1}^{n+1}, \\ v_3^{n+1}(x) &= (B_3^n(x) + |B_3^n(x)|) C_{i,j}^{n+1} + (B_3^n(x) - |B_3^n(x)|) C_{i+1,j}^{n+1}, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} k_1(x) &= \frac{m(s_1)}{h_1} \left(\frac{1}{h_1} \int_{x_{i,j}}^{x_{i,j+1}} \frac{ds}{D(s)} \right)^{-1}, & B_1^n(x) &= \frac{m(s_1)}{2} U_{1,i,j+1/2}^n, \\ k_2(x) &= \frac{m(s_2)}{h_2} \left(\frac{1}{h_2} \int_{x_{i,j}}^{x_{i-1,j-1}} \frac{ds}{D(s)} \right)^{-1}, & B_2^n(x) &= \frac{m(s_2)}{2} U_{2,i-1/2,j-1/2}^n, \\ k_3(x) &= \frac{m(s_3)}{h_3} \left(\frac{1}{h_3} \int_{x_{i,j}}^{x_{i+1,j}} \frac{ds}{D(s)} \right)^{-1}, & B_3^n(x) &= \frac{m(s_3)}{2} U_{3,i+1/2,j}^n. \end{aligned}$$

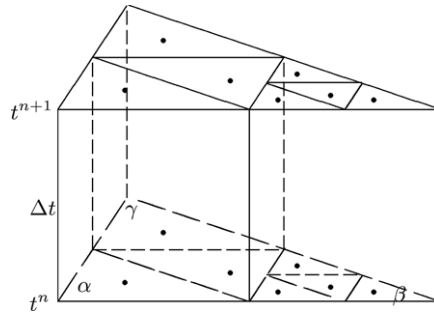
Then we can obtain the following triangular cell-centered difference scheme of (2.1)(c):

$$\begin{aligned} (C^{n+1} - C^n) \int_e \phi(x) dx + \Delta t \sum_{l=1}^3 (w_l^{n+1}(x) + v_l^{n+1}(x)) &= \bar{c}(x, t^{n+1}, C^{n+1}) \int \int_{V \times [t^n, t^{n+1}]} q(x, t) dx dt \\ &\equiv \bar{c}(x, t^{n+1}, C^{n+1}) \varphi^{n+1}, \end{aligned} \quad (4.8)$$

where $\varphi^{n+1} = \int \int_{V \times [t^n, t^{n+1}]} q(x, t) dx dt$. For triangles that intersect $\partial \Omega$, we suppose that $c(x, t)$ is extended continuously in the whole triangle e . Similarly for regular fine grids $x \in \omega_2$, we also have the scheme (4.8).

For an interface FV $e \times [t^n, t^{n+1}]$, we consider the case shown in Fig. 4, where the distribution of grids at each time level is as shown in Fig. 3. When we apply the scheme (4.8) at time t^{n+1} ($n = 1, 2, \dots, J$), we must obtain values of the grid function at the nodes Q_4^* and Q_5^* at time t^{n+1} by interpolation. Here we use piecewise constant or linear interpolation in space. At time $t = t^{n+1}$, for $x = Q_4^*$ and $x = Q_5^*$,

$$\begin{aligned} \int_{t^n}^{t^{n+1}} \int_{s_1} W_{v_1} ds dt &= \int_{t^n}^{t^{n+1}} \left[\int_{s'_1} W_{v_1} ds + \int_{s''_1} W_{v_1} ds \right] dt, \\ \int_{t^n}^{t^{n+1}} \int_{s_1} V_{v_1} ds dt &= \int_{t^n}^{t^{n+1}} \left[\int_{s'_1} V_{v_1} ds + \int_{s''_1} V_{v_1} ds \right] dt \end{aligned}$$

Fig. 4. Interface triangle $e = e(x)$.

are approximated by

$$\begin{aligned}\Delta t w_1^{n+1}(Q_0) &= -\Delta t w_1^{n+1}(Q_4) - \Delta t w_1^{n+1}(Q_5) \\ &= -\Delta t \frac{k_1(Q_4)}{1 + |B_1^n(Q_4)|/k_1(Q_4)} (C^{n+1}(Q_4) - C^{n+1}(Q_4^*)) \\ &\quad - \Delta t \frac{k_1(Q_5)}{1 + |B_1^n(Q_5)|/k_1(Q_5)} (C^{n+1}(Q_5) - C^{n+1}(Q_5^*)), \\ \Delta t v_1^{n+1}(Q_0) &= -\Delta t v_1^{n+1}(Q_4) - \Delta t v_1^{n+1}(Q_5) \\ &= -\Delta t [(B_1^n(Q_4) + |B_1^n(Q_4)|)C^{n+1}(Q_4) + (B_1^n(Q_4) - |B_1^n(Q_4)|)C^{n+1}(Q_4^*)] \\ &\quad - \Delta t [(B_1^n(Q_5) + |B_1^n(Q_5)|)C^{n+1}(Q_5) + (B_1^n(Q_5) - |B_1^n(Q_5)|)C^{n+1}(Q_5^*)],\end{aligned}$$

where

$$\begin{aligned}k_1(Q_4) &= \frac{m(s'_1)}{h_f} \left(\frac{1}{h_f} \int_{Q_4}^{Q_4^*} \frac{ds}{D(s)} \right)^{-1}, \quad k_1(Q_5) = \frac{m(s''_1)}{h_f} \left(\frac{1}{h_f} \int_{Q_5}^{Q_5^*} \frac{ds}{D(s)} \right)^{-1}, \\ B_1^n(Q_4) &= \frac{m(s'_1)}{2} U_{1, \frac{Q_0+Q_1}{2}}^n, \quad B_1^n(Q_5) = \frac{m(s''_1)}{2} U_{1, \frac{Q_0+Q_1}{2}}^n.\end{aligned}$$

If we use piecewise constant interpolation to approximate values of the grid function at the nodes Q_4^* and Q_5^* , $C^{n+1}(P_4^*) = C^{n+1}(P_5^*) = C^{n+1}(P_0)$. Then

$$\begin{aligned}w_1^{n+1}(Q_4) &= \frac{k_1(Q_4)}{1 + |B_1^n(Q_4)|/k_1(Q_4)} \Delta_1 C^{n+1}(Q_4), \\ w_1^{n+1}(Q_5) &= \frac{k_1(Q_5)}{1 + |B_1^n(Q_5)|/k_1(Q_5)} \Delta_1 C^{n+1}(Q_5), \\ v_1^{n+1}(Q_4) &= (B_1^n(Q_4) + |B_1^n(Q_4)|)C^{n+1}(Q_4) + (B_1^n(Q_4) - |B_1^n(Q_4)|)C^{n+1}(Q_0), \\ v_1^{n+1}(Q_5) &= (B_1^n(Q_5) + |B_1^n(Q_5)|)C^{n+1}(Q_5) + (B_1^n(Q_5) - |B_1^n(Q_5)|)C^{n+1}(Q_0).\end{aligned}\tag{4.9}$$

If we use piecewise linear interpolation in space,

$$\begin{aligned}w_1^{n+1}(Q_4) &= \frac{k_1(Q_4)}{1 + |B_1^n(Q_4)|/k_1(Q_4)} \Delta_1 C^{n+1}(Q_4) - \frac{1}{2} \cdot \frac{k_1(Q_4)}{1 + |B_1^n(Q_4)|/k_1(Q_4)} \Delta_3 C^{n+1}(Q_0), \\ w_1^{n+1}(Q_5) &= \frac{k_1(Q_5)}{1 + |B_1^n(Q_5)|/k_1(Q_5)} \Delta_1 C^{n+1}(Q_5) - \frac{1}{2} \cdot \frac{k_1(Q_5)}{1 + |B_1^n(Q_5)|/k_1(Q_5)} \Delta_2 C^{n+1}(Q_0), \\ v_1^{n+1}(Q_4) &= (B_1^n(Q_4) + |B_1^n(Q_4)|)C^{n+1}(Q_4) + (B_1^n(Q_4) - |B_1^n(Q_4)|)C^{n+1}(Q_0) \\ &\quad + \frac{1}{2} (B_1^n(Q_4) - |B_1^n(Q_4)|) \Delta_3 C^{n+1}(Q_0),\end{aligned}$$

$$v_1^{n+1}(Q_5) = (B_1^n(Q_5) + |B_1^n(Q_5)|)C^{n+1}(Q_5) + (B_1^n(Q_5) - |B_1^n(Q_5)|)C^{n+1}(Q_0) + \frac{1}{2}(B_1^n(Q_5) - |B_1^n(Q_5)|) \Delta_2 C^{n+1}(Q_0). \quad (4.10)$$

In the following convergence analysis, we only consider the case of linear interpolation.

5. Convergence analysis

Define the error of difference schemes constructed in Section 4

$$\pi(x, t^n) = P^n(x) - p(x, t^n), \quad \varepsilon(x, t^n) = C^n(x) - c(x, t^n), \quad x \in \omega. \quad (5.1)$$

First consider the difference scheme (4.1)–(4.3) of the pressure equation (2.1)(a) on composite grids. Substituting $P^n(x) = \pi^n(x) + p^n(x)$ into (4.2),

$$\sum_{l=1}^3 \bar{w}_l^n(x) = \bar{\varphi}^n(x) - \sum_{l=1}^3 \bar{w}_l^n(x) = \bar{\psi}^n, \quad x \in \omega,$$

where the fluxes on the left are evaluated by the values of π and the fluxes on the right by the values of p . Put $\bar{\psi}^n$ into the following form:

$$\bar{\psi}^n = \sum_{l=1}^3 \left\{ \int_{s_l} W_{v_l} ds - w_l^n(x) \right\} \equiv \sum_{l=1}^3 \bar{\eta}_l^n(x).$$

Similarly to the deduction of Theorem 4.1 in [7], we can obtain

$$\|\pi^n\|_{1,\omega} \leq K \sum_{l=1}^3 \|\bar{\eta}_l^n\|_{0,\omega},$$

then we can see that in order to estimate π^n , we only need to estimate $\bar{\eta}_l^n$.

For regular grids ($l = 1$), we have

$$\begin{aligned} \bar{\eta}_1(Q_0) &= - \int_{s_1} a(x, c^n) \frac{\partial p}{\partial v_1} ds - \bar{k}_1^n(Q_0)[p(Q_0) - p(Q_1)] \\ &= - \int_{s_1} [a(x, c^n) - a(x, C_{i,j+1/2,k}^n)] \frac{\partial p}{\partial v_1} ds - \int_{s_1} a(x, C_{i,j+1/2,k}^n) \frac{\partial p}{\partial v_1} ds - \bar{k}_1^n(Q_0)[p(Q_0) - p(Q_1)]. \end{aligned}$$

Because of the Lipschitz condition of $a(x, c)$ and the Bramble–Hilbert lemma, we can obtain

$$\|\pi^n\|_{1,\omega} \leq K(\|p^n\|_{3,\infty})(\|\varepsilon^n\| + h^2).$$

Similarly for irregular grids, $\|\pi^n\|_{1,\omega} \leq K(\|\varepsilon^n\| + h^{3/2})$. Therefore

$$\|\pi^n\|_{1,\omega} \leq K(\|p^n\|_{3,\infty})(\|\varepsilon^n\| + h^{3/2}). \quad (5.2)$$

Then we study the difference scheme (4.7), (4.8) and (4.10) of the saturation equation (2.1)(c) on composite grids. The local refinement in space is shown in Fig. 5. We multiply (4.8) by $C^{n+1}(x)$ and sum over $x \in \omega_1$, then

$$\begin{aligned} &\sum_{x \in \omega_1} \left\{ (C^{n+1}(x) - C^n(x)) \int_e \phi(x) dx + \Delta t \sum_{l=1}^3 (w_l^{n+1}(x) + v_l^{n+1}(x)) \right\} C^{n+1}(x) \\ &= \frac{1}{2} \sum_{x \in \omega_1} [(C^{n+1}(x))^2 - (C^n(x))^2 + (C^{n+1}(x) - C^n(x))^2] \int_e \phi(x) dx + \Delta t \sum_{l=1}^3 (I_{1,l}^{n+1} + J_{1,l}^{n+1}), \end{aligned} \quad (5.3)$$

For regular grids $x \in \omega$, using partial summation, we have

$$\begin{aligned}\Delta t J_{1,1}^{n+1} + \Delta t J_{2,1}^{n+1} &= \Delta t \sum_{x \in \omega_1 \setminus S_1} \alpha_1(x) |B_1^n(x)| (\Delta_1 C^{n+1}(x))^2 \\ &\quad + \Delta t \sum_{x \in \omega_2} \alpha_1(x) |B_1^n(x)| (\Delta_1 C^{n+1}(x))^2 + \Delta t \sum_{x \in \omega} B_1^n(x) (C^{n+1}(x))^2, \\ \Delta t J_{1,2}^{n+1} + \Delta t J_{2,2}^{n+1} &= \Delta t \sum_{x \in \omega_1 \setminus S_2} \alpha_2(x) |B_2^n(x)| (\Delta_2 C^{n+1}(x))^2 \\ &\quad + \Delta t \sum_{x \in \omega_2} \alpha_2(x) |B_2^n(x)| (\Delta_2 C^{n+1}(x))^2 + \Delta t \sum_{x \in \omega} B_2^n(x) (C^{n+1}(x))^2, \\ \Delta t J_{1,3}^{n+1} + \Delta t J_{2,3}^{n+1} &= \Delta t \sum_{x \in \omega} \alpha_3(x) |B_3^n(x)| (\Delta_3 C^{n+1}(x))^2 + \Delta t \sum_{x \in \omega} B_3^n(x) (C^{n+1}(x))^2.\end{aligned}\tag{5.8}$$

Adding (5.3) and (5.4) and taking into account the equalities (5.5)–(5.8), we see that

$$\begin{aligned}\sum_{x \in \omega} \left\{ (C^{n+1}(x) - C^n(x)) \int_e \phi(x) dx + \Delta t \sum_{l=1}^3 (w_l^{n+1}(x) + v_l^{n+1}(x)) \right\} C^{n+1}(x) \\ = \frac{1}{2} \sum_{x \in \omega} ((C^{n+1}(x))^2 - (C^n(x))^2) \int_e \phi(x) dx + \frac{1}{2} \sum_{x \in \omega} (C^{n+1}(x) - C^n(x))^2 \int_e \phi(x) dx \\ + \sum_{l=1}^3 \sum_{x \in \omega} \Delta t (\tilde{\beta}_l^n(x) + |\tilde{B}_l^n(x)|) (\Delta_l C^{n+1}(x))^2 + \sum_{l=1}^3 \sum_{x \in \omega} \Delta t B_l^n(x) (C^{n+1}(x))^2,\end{aligned}$$

where $\tilde{\beta}_l^n(x) = \alpha_l(x) \beta_l^n(x)$, $\beta_l^n(x) = \frac{k_l(x)}{1 + |B_l^n(x)|/k_l(x)}$, $\tilde{B}_l^n(x) = \alpha_l(x) B_l^n(x)$, the summation $\sum_{x \in \omega}$ of the third term in the right-hand side is taken to mean over those grid points $x \in \omega$, for which $\Delta_l C(x)$ is defined ($l = 1, 2, 3$).

We need an induction hypothesis: there exist constants $M_1 > 0$ and $M_2 > 0$ so that

$$\begin{aligned}\text{(a)} \quad |U_l^n(x)| &\leq M_1, \quad n = 0, 1, \dots, J, \quad l = 1, 2, 3, \quad x \in \omega, \\ \text{(b)} \quad |\bar{c}(x, t^n, C^n)| &\leq M_2, \quad n = 0, 1, \dots, J, \quad x \in \omega.\end{aligned}\tag{5.9}$$

Let A_0 be the matrix in the case of using piecewise constant interpolation,

$$\begin{aligned}(A_0 \mathbf{C}^{n+1}, \mathbf{C}^{n+1}) &= \frac{1}{2} \sum_{x \in \omega} ((C^{n+1}(x))^2 - (C^n(x))^2) \int_e \phi(x) dx + \frac{1}{2} \sum_{x \in \omega} (C^{n+1}(x) - C^n(x))^2 \int_e \phi(x) dx \\ &\quad + \sum_{l=1}^3 \sum_{x \in \omega} \Delta t (\tilde{\beta}_l^n(x) + |\tilde{B}_l^n(x)|) (\Delta_l C^{n+1}(x))^2 + \sum_{l=1}^3 \sum_{x \in \omega} \Delta t B_l^n(x) (C^{n+1}(x))^2.\end{aligned}\tag{5.10}$$

Summing over $n = 1, \dots, J-1$, and using the hypothesis (5.9), the difference scheme defined by (4.7)–(4.9) is stable and the following prior estimate holds:

$$\sum_{n=0}^{J-1} (A_0 \mathbf{C}^{n+1}, \mathbf{C}^{n+1}) \leq K \sum_{n=0}^{J-1} \sum_{\omega} (\varphi^{n+1}(x))^2,\tag{5.11}$$

where K is a constant independent of h and Δt . Let A be the matrix in the case of using piecewise linear interpolation, then applying the Cauchy inequality and the inequality $-|ab| \geq -\frac{a^2+b^2}{2}$, we can obtain

$$\gamma_1(A_0 \mathbf{C}^{n+1}, \mathbf{C}^{n+1}) \leq (A \mathbf{C}^{n+1}, \mathbf{C}^{n+1}) \leq \gamma_2(A_0 \mathbf{C}^{n+1}, \mathbf{C}^{n+1}).\tag{5.12}$$

Using (5.11) and (5.12), we have the following theorem:

Theorem 5.1. *The finite difference scheme defined by (4.7), (4.8) and (4.10) is stable and the following estimate holds:*

$$\sum_{n=0}^{J-1} (A\mathbf{C}^{n+1}, \mathbf{C}^{n+1}) \leq K \sum_{n=0}^{J-1} \sum_{\omega} (\varphi^{n+1}(x))^2, \quad (5.13)$$

where K is a constant independent of h and Δt .

Substituting $C^n(x) = \varepsilon^n(x) + c^n(x)$ into (4.8), we have

$$\begin{aligned} & (\varepsilon^{n+1} - \varepsilon^n) \int_{e(x)} \phi(x) dx + \Delta t \sum_{l=1}^3 (w_l^{n+1} + v_l^{n+1}) \\ &= \bar{c}(x, t^{n+1}, C^{n+1}) \varphi^{n+1} - (c^{n+1} - c^n) \int_{e(x)} \phi(x) dx - \Delta t \sum_{l=1}^3 (w_l^{n+1} + v_l^{n+1}) \\ &\equiv \psi_1^{n+1}, \quad x \in \omega_1, \end{aligned} \quad (5.14)$$

$$\begin{aligned} & (\varepsilon^{n+1} - \varepsilon^n) \int_{e(x)} \phi(x) dx + \Delta t \sum_{l=1}^3 (w_l^{n+1} + v_l^{n+1}) \\ &= \bar{c}(x, t^{n+1}, C^{n+1}) \varphi^{n+1} - (c^{n+1} - c^n) \int_{e(x)} \phi(x) dx - \Delta t \sum_{l=1}^3 (w_l^{n+1} + v_l^{n+1}) \\ &\equiv \psi_2^{n+1}, \quad x \in \omega_2, \end{aligned} \quad (5.15)$$

where the fluxes on the left are evaluated by the values of ε and the fluxes on the right by the values of c . Because φ^{n+1} satisfies (4.8), we can write ψ_1^{n+1} and ψ_2^{n+1} into

$$\begin{aligned} \psi_1^{n+1} &= \int_{t^n}^{t^{n+1}} \left\{ \int_e \phi(\xi) \frac{\partial c}{\partial t}(\xi, t) d\xi - \int_e \phi(x) dx \cdot \frac{\partial c}{\partial t}(x, t) \right\} dt \\ &+ \sum_{l=1}^3 \left\{ \int_{t^n}^{t^{n+1}} \int_{s_l} W_{v_l} ds dt - \Delta t w_l^{n+1} \right\} + \sum_{l=1}^3 \left\{ \int_{t^n}^{t^{n+1}} \int_{s_l} V_{v_l} ds dt - \Delta t v_l^{n+1} \right\} \\ &\equiv \int_{e(x)} \phi(x) dx \cdot \Delta t \eta_{1,0}^{n+1}(x) + \Delta t \sum_{l=1}^3 \eta_{1,l}^{n+1}(x) + \Delta t \sum_{l=1}^3 \mu_{1,l}^{n+1}(x), \quad x \in \omega_1, \end{aligned} \quad (5.16)$$

$$\begin{aligned} \psi_2^{n+1} &= \int_{t^n}^{t^{n+1}} \left\{ \int_e \phi(\xi) \frac{\partial c}{\partial t}(\xi, t) d\xi - \int_e \phi(x) dx \cdot \frac{\partial c}{\partial t}(x, t) \right\} dt \\ &+ \sum_{l=1}^3 \left\{ \int_{t^n}^{t^{n+1}} \int_{s_l} W_{v_l} ds dt - \Delta t w_l^{n+1} \right\} + \sum_{l=1}^3 \left\{ \int_{t^n}^{t^{n+1}} \int_{s_l} V_{v_l} ds dt - \Delta t v_l^{n+1} \right\} \\ &\equiv \int_{e(x)} \phi(x) dx \cdot \Delta t \eta_{2,0}^{n+1}(x) + \Delta t \sum_{l=1}^3 \eta_{2,l}^{n+1}(x) + \Delta t \sum_{l=1}^3 \mu_{2,l}^{n+1}(x), \quad x \in \omega_2. \end{aligned} \quad (5.17)$$

Note that the error ε satisfies

$$\begin{aligned} \varepsilon^0(x) &= 0, \quad x \in \omega, \\ \varepsilon^{n+1}(x) &= 0, \quad x \in \partial\Omega, n = 0, \dots, J-1. \end{aligned} \quad (5.18)$$

We multiply (5.14) and (5.15) by $\varepsilon^{n+1}(x)$, and sum over ω_1 and ω_2 respectively. Taking into account (5.10) and (5.12) and properties that ϕ and D satisfy, we have

$$\begin{aligned} & \sum_{\omega} \int_{\omega} \phi(x) dx ((\varepsilon^{n+1}(x))^2 - (\varepsilon^n(x))^2) + K \sum_{l=1}^3 \left\{ \sum_{\omega} \Delta t [(\Delta_l \varepsilon^{n+1})^2] \right\} \\ & \leq 2 \sum_{\omega_1} \psi_1^{n+1} \varepsilon^{n+1} + 2 \sum_{\omega_2} \psi_2^{n+1} \varepsilon^{n+1}. \end{aligned} \quad (5.19)$$

Similarly to the above,

$$\begin{aligned} \sum_{\omega_1} \psi_1^{n+1} \varepsilon^{n+1} + \sum_{\omega_2} \psi_2^{n+1} \varepsilon^{n+1} &= \sum_{\omega_1} \int_{\omega_1} \phi(x) dx \Delta t \eta_{1,0}^{n+1} \varepsilon^{n+1} + \sum_{\omega_2} \int_{\omega_2} \phi(x) dx \Delta t \eta_{2,0}^{n+1} \varepsilon^{n+1} \\ &+ \Delta t \sum_{l=1}^3 \left[\sum_{\omega_1} \alpha_l(x) (\eta_{1,l}^{n+1} + \mu_{1,l}^{n+1}) \Delta_l \varepsilon^{n+1} \right. \\ &\left. + \sum_{\omega_2} \alpha_l(x) (\eta_{2,l}^{n+1} + \mu_{2,l}^{n+1}) \Delta_l \varepsilon^{n+1} \right]. \end{aligned} \quad (5.20)$$

Applying the Cauchy inequality to the right-hand side of this identity and using the simplest “imbedding” inequality [9], we can obtain

$$\begin{aligned} \sum_{x \in \omega_1} \psi_1^{n+1} \varepsilon^{n+1} + \sum_{x \in \omega_2} \psi_2^{n+1} \varepsilon^{n+1} &\leq K \left\{ \sum_{\omega_1} \Delta t \left[h_c^2 (\eta_{1,0}^{n+1})^2 + \sum_{l=1}^3 ((\eta_{1,l}^{n+1})^2 + (\mu_{1,l}^{n+1})^2) \right] \right. \\ &+ \sum_{\omega_2} \Delta t \left[h_f^2 (\eta_{2,0}^{n+1})^2 + \sum_{l=1}^3 ((\eta_{2,l}^{n+1})^2 + (\mu_{2,l}^{n+1})^2) \right] \Bigg\}^{\frac{1}{2}} \\ &\times \left\{ \sum_{l=1}^3 \left[\sum_{\omega} \Delta t (\Delta_l \varepsilon^{n+1})^2 \right] \right\}^{\frac{1}{2}}. \end{aligned} \quad (5.21)$$

Let us introduce the norm

$$\|y\|^2 \doteq \max_{0 \leq n \leq J} \sum_{\omega} h^2 (y^n(x))^2 + \sum_{n=0}^{J-1} \sum_{l=1}^3 \left\{ \sum_{x \in \omega} \Delta t [(\Delta_l y^{n+1}(x))^2] \right\}. \quad (5.22)$$

We substitute (5.21) into (5.19), use the ε -inequality, sum over $n = 0, 1, \dots, J-1$ and obtain the following theorem:

Theorem 5.2. *The error ε of the difference scheme defined by (4.7), (4.8) and (4.10) satisfies the estimate*

$$\begin{aligned} \|\varepsilon\|^2 &\leq K \sum_{n=0}^{J-1} \left\{ \sum_{\omega_1} \Delta t \left[h_c^2 (\eta_{1,0}^{n+1})^2 + \sum_{l=1}^3 ((\eta_{1,l}^{n+1})^2 + (\mu_{1,l}^{n+1})^2) \right] \right. \\ &\left. + \sum_{\omega_2} \Delta t \left[h_f^2 (\eta_{2,0}^{n+1})^2 + \sum_{l=1}^3 ((\eta_{2,l}^{n+1})^2 + (\mu_{2,l}^{n+1})^2) \right] \right\}, \end{aligned} \quad (5.23)$$

where $\eta_{1,l}^n, \eta_{2,l}^n, \mu_{1,l}^n, \mu_{2,l}^n, l = 1, 2, 3$ and $\eta_{1,0}^n, \eta_{2,0}^n$ are local truncation errors defined by (5.16) and (5.17), K is a constant independent of h and Δt .

From Theorem 5.2, we see that in order to estimate ε , we only need to estimate the terms in the right-hand side in (5.23). In the following, we suppose the solution $c(x, t)$ satisfies the condition (2.4) and estimate these terms.

Using the Bramble–Hilbert lemma and the Hölder inequality, we can estimate $\eta_{2,0}^{n+1}(x)$ and $\eta_{2,0}^{n+1}(x)$. Then, the estimate holds:

$$\left\{ \sum_{n=0}^{J-1} \left[\sum_{\omega_1} \Delta t h_c^2 (\eta_{1,0}^{n+1})^2 + \sum_{\omega_2} \Delta t h_f^2 (\eta_{2,0}^{n+1})^2 \right] \right\}^{1/2} \leq K h_c^2 \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_1))} + K h_f^2 \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_2))}. \quad (5.24)$$

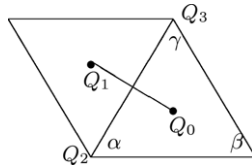


Fig. 6. Regular triangles $e = e(x)$.

In the case of regular coarse grid points (see Fig. 6), by the Bramble–Hilbert lemma we obtain

$$\sum_{n=0}^{J-1} \sum_{\omega_1} \Delta t (\eta_{1,1}^{n+1})^2 \leq K h_c^4 \max_{0 \leq t \leq T} \|c\|_{H^3(\Omega_1)}^2 + K \Delta t^2 \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_1))}^2. \quad (5.25)$$

Split the sum \sum_{ω_2} into two sums: $\sum_{\omega_2^{(1)}}$ over all regular points and $\sum_{\omega_2^{(2)}}$ over the irregular points in ω_2 . Similarly we have

$$\sum_{n=0}^{J-1} \sum_{\omega_2^{(1)}} \Delta t (\eta_{2,1}^{n+1})^2 \leq K h_f^4 \max_{0 \leq t \leq T} \|c\|_{H^3(\Omega_2)}^2 + K \Delta t^2 \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_2))}^2. \quad (5.26)$$

We can also obtain similar estimates on $\eta_{1,l}^{n+1}$ and $\eta_{2,l}^{n+1}$ ($l = 2, 3$). Then

$$\left\{ \sum_{n=0}^{J-1} \left[\sum_{\omega_1} \Delta t \sum_{l=1}^3 (\eta_{1,l}^{n+1})^2 + \sum_{\omega_2^{(1)}} \Delta t \sum_{l=1}^3 (\eta_{2,l}^{n+1})^2 \right] \right\}^{1/2} \leq K \left\{ \Delta t \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_1))} + \Delta t \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_2))} + h_c^2 \max_{0 \leq t \leq T} \|c\|_{H^3(\Omega_1)} + h_f^2 \max_{0 \leq t \leq T} \|c\|_{H^3(\Omega_2)} \right\}. \quad (5.27)$$

For $\mu_{1,1}^{n+1}(Q_0)$, we need an induction hypothesis: Let $\Delta t = O(h)$, we assume that

$$\|\varepsilon^n\|_{0,\omega} \leq K h. \quad (5.28)$$

Then using the induction hypothesis (5.28) and the estimate (5.2), we have

$$|\mu_{1,1}^{n+1}(Q_0)| \leq K h_c^2 \max_{t^n \leq t \leq t^{n+1}} \|c\|_{H^2(\bar{\omega})} + K (\Delta t)^{1/2} \left(\int_{t^n}^{t^{n+1}} \left\| \frac{\partial c}{\partial t} \right\|_{H^2(\bar{\omega})}^2 dt \right)^{1/2}.$$

Similarly we can estimate $\mu_{2,1}^{n+1}$, $\mu_{1,l}^{n+1}$ and $\mu_{2,l}^{n+1}$ ($l = 2, 3$). Therefore

$$\left\{ \sum_{n=0}^{J-1} \left[\sum_{\omega_1} \Delta t \sum_{l=1}^3 (\mu_{1,l}^{n+1})^2 + \sum_{\omega_2^{(1)}} \Delta t \sum_{l=1}^3 (\mu_{2,l}^{n+1})^2 \right] \right\}^{1/2} \leq K \left\{ \Delta t \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_1))} + \Delta t \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_2))} + h_c^2 \max_{0 \leq t \leq T} \|c\|_{H^2(\Omega_1)} + h_f^2 \max_{0 \leq t \leq T} \|c\|_{H^2(\Omega_2)} \right\}. \quad (5.29)$$

Next, we consider irregular grids (see Fig. 4). Let Ω_δ be a strip with δ between Ω_1 and Ω_2 , and using the Bramble–Hilbert lemma, we can obtain estimates on $\eta_{2,l}^{n+1}$ and $\mu_{2,l}^{n+1}$ ($l = 1, 2, 3$). Then

$$\left\{ \sum_{n=0}^{J-1} \sum_{\omega_2^{(2)}} \left[\Delta t \sum_{l=1}^3 (\eta_{2,l}^{n+1})^2 + \Delta t \sum_{l=1}^3 (\mu_{2,l}^{n+1})^2 \right] \right\}^{1/2} \leq K h_f \max_{0 \leq t \leq T} \|c\|_{H^2(\Omega_\delta)} + K \Delta t \left\| \frac{\partial c}{\partial t} \right\|_{L^2(H^2(\Omega_\delta))}. \quad (5.30)$$

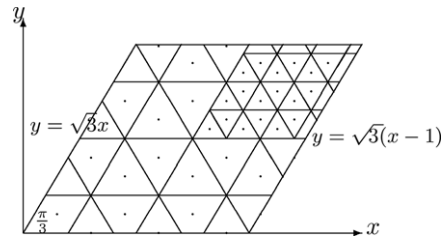


Fig. 7. The composite grid.

Substituting the estimates (5.24), (5.27), (5.29) and (5.30) into (5.23), we can obtain

$$\|c - C\| \leq K \left\{ (h_c^2 + \Delta t) \|c\|_{\Omega_1} + (h_f^2 + \Delta t) \|c\|_{\Omega_2} + (h_f + \Delta t) \|c\|_{\Omega_\delta} \right\}, \quad (5.31)$$

where the norm $\|\cdot\|$ is defined by (5.22), $\|u\|_{\Omega_1}$, $\|u\|_{\Omega_2}$ and $\|u\|_{\Omega_\delta}$ are certain norms of the solution c which are defined according to (5.24), (5.27), (5.29) and (5.30).

It remains to check induction hypotheses (5.9) and (5.28). When $n = 0$, according to the initial condition (2.2), it is easy to obtain (5.9)(a) and (5.28). Next we suppose that (5.9)(a) and (5.28) hold when $1 \leq n \leq N$. Then if $\Delta t = O(h)$, using (5.2) and (5.31), we can obtain (5.9)(a) and (5.28) when $n = N + 1$. And using the definition of \bar{c} and (5.31), we can obtain (5.9)(b). So the induction hypotheses (5.9) and (5.28) hold, then we have the following theorem:

Theorem 5.3. *Let the exact solutions p , u and c of (2.1)–(2.3) satisfy the condition (2.4), then the error estimate (5.31) holds when $\Delta t = O(h)$.*

6. Numerical results

Consider the following parabolic problem

$$\frac{\partial c}{\partial t} + \nabla \cdot (-\nabla c(x, t) + c(x, t)) = f(x, t), \quad x \in \Omega, t > 0, \quad (6.1)$$

where $\Omega = \{(x_1, x_2) : 0 \leq x_2 \leq \frac{\sqrt{3}}{2}, x_2/\sqrt{3} \leq x_1 \leq x_2/\sqrt{3} + 1\}$. Let the time step $\Delta t = 0.1$ and the discretizations in Ω be done as shown in Fig. 7, with $h = 1/(n - \frac{2}{3})$ for some given integer n and ratio between coarse-grid and fine-grid cell sizes equal to m , $m \geq 1$, integer. Choose the refined subregion

$$\Omega_2 = \left\{ (x_1, x_2) : x_2^0 < x_2 \leq \frac{\sqrt{3}}{2}, x_1^0 + (x_2 - x_2^0)/\sqrt{3} < x_1 \leq x_2/\sqrt{3} + 1 \right\},$$

where $x_1^0 = (i_0 + (j_0 - 1)/2)h$, $x_2^0 = \frac{\sqrt{3}}{2}(j_0 - \frac{1}{3})h$, $0 < i_0 < n$, $0 < j_0 < n$. Denote by C the numerical approximations to $c(x, t)$ and assume that $i_0 = j_0 = n/2$, $n = 4$, $m = 2$.

Choose $f(x, t)$ such that the exact solution of (6.1) is

$$c(x, t) = \frac{1}{10} \exp(t^2)x_2 \left(x_2 - \frac{\sqrt{3}}{2} \right) (x_2 - \sqrt{3}x_1)(x_2 - \sqrt{3}(x_1 - 1)) \exp((x_1 + x_2)^2). \quad (6.2)$$

When $t = 0.2$, the exact solution is shown in Fig. 8.

We compare the numerical results obtained by the scheme without local refinement and with local refinement. When $t = 0.2$, the computational results are as in Figs. 9 and 10.

Numerical results of the maximal absolute error are shown in Table 1.

The numerical results show that in the case of highly localized properties, using the local refinement technique, we can obtain efficient numerical approximations. These results also indicate that the method proposed in this paper can be widely applied to some application fields, such as energy numerical simulation and environmental science.

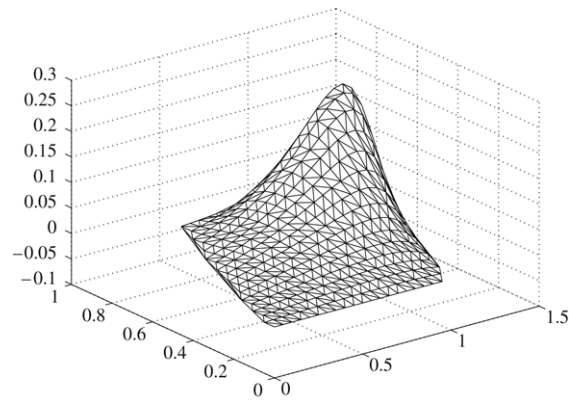


Fig. 8. The exact solution of (6.1).

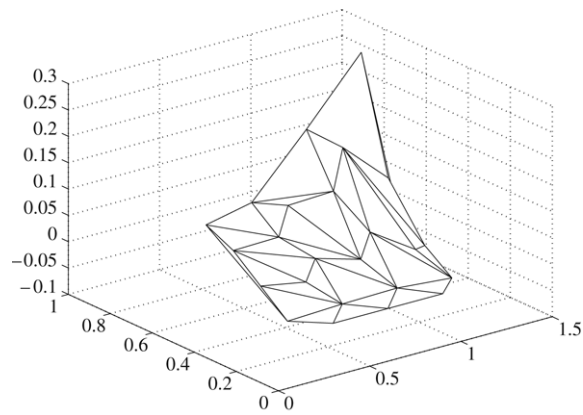


Fig. 9. Results obtained by the scheme without local refinement.

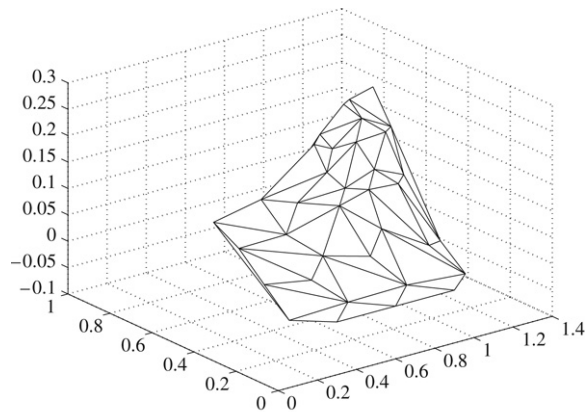


Fig. 10. Results obtained by the scheme with local refinement.

Table 1
The maximal absolute error

/	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
Without local refinement	0.058699	0.096790	0.122481	0.141203	0.156624
With local refinement	0.007418	0.015356	0.024495	0.035454	0.048916

7. Conclusion

In this paper, we have considered the finite difference scheme of a system of two nonlinear partial differential equations of describing the miscible displacement flow in porous media on composite grids with local refinement in space. Using the FV method, the finite difference schemes of the coupled equations on composite triangular cell-centered grids are constructed. Studying their stability and convergence properties, the error estimate in the energy norm is obtained. Moreover, we give a numerical example to illustrate the theory obtained above. The local refinement technique in this paper can be extended to the case of a nonuniform grid and to the three-dimensional case. These results are of great importance for the research on numerical simulation of the fluid flow problem.

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